The Bohr model

An electron orbits a proton in a circular orbit of radius r.

(a) Show using ideas of classical physics as in Topic D that total energy of the electron-proton system is given by $E = -\frac{ke^2}{2r}$.

According to the Bohr model for hydrogen the quantization condition $m_e v_n r_n = n\frac{h}{2\pi}$ leads to $r_n = a_0 n^2$ where $a_0 = \frac{h^2}{4\pi^2 ke^2 m_e}$ and n is an integer.

(b) Show that in the Bohr model, for the electron in the n^{th} state

(i) the energy of the atom is given by $E_n = -\frac{ke^2}{2a_0}\frac{1}{n^2}$,

(ii) the speed of the electron is given by $v_n = \frac{h}{2\pi m_e a_0}\frac{1}{n}$,

(iii) the frequency of revolution of the electron is given by $f_n = \frac{h}{4\pi^2 m_e a_0^2}\frac{1}{n^3}$.

(c) An electron is in the $(n+1)^{st}$ state of a hydrogen atom. The electron makes a transition to the state n.

(i) Convince yourself that if n is a very large integer compared to 1 then $\frac{1}{n^2} - \frac{1}{(n+1)^2} \approx \frac{2}{n^3}$.

(ii) Hence deduce the frequency of the photon emitted in the transition from state $(n+1)$ to state n.

(iii) Comment on your answer.

Answers

(a) $E=\frac{1}{2}m_e v^2-\frac{ke^2}{r}$. But from Newton's second law: $\frac{m_e v^2}{r}=\frac{ke^2}{r^2}$, hence $m_e v^2=\frac{ke^2}{r}$.

Thus, $E=\frac{ke^2}{2r}-\frac{ke^2}{r}=-\frac{ke^2}{2r}$.

(b)

(i) Since $r_n=a_0 n^2$ from (a) $E_n=-\frac{ke^2}{2a_0}\frac{1}{n^2}$.

(ii) From $m_e v_n r_n=n\frac{h}{2\pi}$, we have $v_n=n\frac{h}{2\pi m_e r}=n\frac{h}{2\pi m_e a_0 n^2}=\frac{h}{2\pi m_e a_0}\frac{1}{n}$.

(iii) $v_n=\frac{2\pi r_n}{T_n}=2\pi r_n f_n \Rightarrow f_n=\frac{v_n}{2\pi r_n}=\frac{\frac{h}{2\pi m_e a_0}\frac{1}{n}}{2\pi a_0 n^2}=\frac{h}{4\pi^2 m_e a_0^2}\frac{1}{n^3}$.

(c) The difference in energy is

(i)

$$\Delta E=\frac{ke^2}{2a_0}\frac{1}{n^2}--\frac{ke^2}{2a_0}\frac{1}{(n+1)^2}$$
$$=\frac{ke^2}{2a_0}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$
$$\approx\frac{ke^2}{2a_0}\frac{2}{n^3}$$
$$=\frac{ke^2}{a_0}\frac{1}{n^3}$$

(ii) Hence $hf=\frac{ke^2}{a_0}\frac{1}{n^3} \Rightarrow f=\frac{ke^2}{ha_0}\frac{1}{n^3}$.

Since $m_e v^2=\frac{ke^2}{r} \Rightarrow ke^2=m_e v^2 r=(m_e vr)v=n\frac{h}{2\pi}\times\frac{h}{2\pi m a_0}\times\frac{1}{n}=\frac{h^2}{4\pi^2 m a_0}$

Hence

$f=\frac{ke^2}{ha_0}\times\frac{1}{n^3}=\frac{1}{ha_0}\times\frac{h^2}{4\pi^2 m a_0}\times\frac{1}{n^3}=\frac{h}{4\pi^2 m_e a_0^2}\times\frac{1}{n^3}$ which is identical to the answer in (b)(iii). This is Bohr's correspondence principle: as the quantum number n becomes large the results of the quantum theory approach those of the classical theory. Here, the photon emitted in the transition from (n+1) to n (quantum result) is the same as the frequency of revolution in the state n. According to classical theory an electron would radiate EM waves of frequency equal to the frequency of revolution.